Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 330 (2007) 729–739
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Periodic weak solutions for a classical one-dimensional isotropic biquadratic Heisenberg spin chain

Boling Guo [a], Ming Zeng [b,*], Fengqiu Su [c]

[a] *Center for Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, PO Box 8009-28, Beijing 100088, China*
[b] *College of Applied Sciences, Beijing University of Technology, PingLe Yuan 100, Chaoyang District, Beijing 100022, China*
[c] *Department of Mathematical and Computer Science, Center University of Nationality, Beijing 100081, China*

Received 22 March 2005
Available online 12 September 2006
Submitted by M.D. Gunzburger

Abstract

In this paper, we consider the model proposed by Lakshmanan, Porsezian and Daniel [M. Lakshmanan, K. Porsezian, M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, Phys. Lett. A 133 (1988) 483–488] describing the continuum isotropic biquadratic Heisenberg spin chain in one dimensional case. Following the results of integrability in [K. Porsezian, M. Daniel, M. Lakshmanan, On the integrability aspects of the one-dimensional classical continuum isotropic Heisenberg spin chain, J. Math. Phys. 33 (5) (1992) 1807–1816], we prove the existence of periodic weak solutions to this model.

Keywords: Isotropic biquadratic Heisenberg spin chain; Weak solutions; Viscosity vanishing method

1. Introduction

In 1988, Lakshmanan, Porsezian and Daniel [1] gave the following Hamiltonian for a classical one-dimensional isotropic biquadratic Heisenberg spin chain with N spins characterized by nearest neighbor bilinear and biquadratic exchange interaction:

$$H = -J \sum_{ij} \left[S_i \cdot S_j + k(S_i \cdot S_j)^2 \right], \quad i, j = 1, 2, \ldots, N. \tag{1.1}$$

* Corresponding author.
E-mail address: mengnan218@sohu.com (M. Zeng).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.08.010

Here the spins S_i's are three-dimensional vectors; J and k denote the bilinear and biquadratic exchange parameters, respectively. The equation of motion corresponding to (1.1) with nearest neighbor interacting spins is

$$\frac{\mathrm{d}S_n}{\mathrm{d}t} = JS_n \times \big\{(S_{n+1} + S_{n-1}) + k\big[S_{n+1}(S_n \cdot S_{n+1}) + S_{n-1}(S_n \cdot S_{n-1})\big]\big\}. \tag{1.2}$$

In order to consider the continuum limit of (1.2) in the long wavelength and low temperature, the authors allow the lattice parameter a between nearest neighbor sites to approach zero and assume a slow variation of S_n over a lattice distance. By introducing $u(x,t) = u(na,t) = S_n(t)$, u can be expanded in a Taylor series up to fourth-order. After rescaling the time variable appropriately, (1.2) takes the form

$$u_t = u \times \left\{u_{xx} + \gamma_1 u_{xxxx} + \gamma_2\left[(u \cdot u_{xx})u_{xx} + \frac{2}{3}(u \cdot u_{xxx})u_x\right]\right\}, \tag{1.3}$$

where $\gamma_1 = a^2/12$ and $\gamma_2 = ka^2/(1+2k)$. When $\gamma_1 = \gamma_2 = 0$, (1.3) turns into the well-known Landau–Lifshitz equation [3]. When $\gamma_1 = 0$ and $\gamma_2 \leqslant 0$, the corresponding equation has been deduced by Fivez [4] when he revisited the one-dimensional classical compressible Heisenberg chain, which was also considered previously by Cieplak and Turski [5]. Furthermore, Guo et al. [6] proved the existence of measure-valued solutions. Since the fourth-order term $S \times S_{xxxx}$ on the right side of (1.3) seems to be dissipative from the mathematical point of view, we want to derive a solution stronger than the one considered in [6], that is just the main result of this paper.

However, the method we will use here cannot be applied for all γ_1 and γ_2, in fact, it is related to the integrability of (1.3). For this purpose, Porsezian et al. [2] turned (1.3) into its equivalent form through a differential geometric approach. It reads

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 2|q|^2 q + \gamma_1 q_{xxxx} - 4\delta_1 |q|^2 q_{xx} - 4\delta_2 |q|^2 q_{xx}^* - 4\delta_3 q|q_x|^2 - 4\delta_4 q^*|q_x|^2 \\ &\quad - 24\delta_5 |q|^4 q = 0, \end{aligned} \tag{1.4}$$

where $\delta_1 = 3\gamma_1 + 2\gamma_2$, $\delta_2 = 2\gamma_1 + \gamma_2$, $\delta_3 = 9\gamma_1 + 4\gamma_2$, $\delta_4 = \frac{7}{2}\gamma_1 + 2\gamma_2$ and $\delta_5 = 3\gamma_1 + \gamma_2/2$. (1.4) can be viewed as a generalization of the nonlinear Schrödinger equation because (1.4) reduces to the nonlinear Schrödinger equation when $\gamma_1 = \gamma_2 = 0$.

Porsezian et al. [2] utilized Painlevé singularity structure analysis to conclude that (1.4) is integrable if and only if $\gamma_2 = -\frac{5}{2}\gamma_1$. Consequently, (1.3) takes the form

$$u_t = u \times \left\{u_{xx} + \gamma_1 u_{xxxx} - \frac{5}{2}\gamma_1\left[(u \cdot u_{xx})u_{xx} + \frac{2}{3}(u \cdot u_{xxx})u_x\right]\right\}, \tag{1.5}$$

Daniel et al. [7] obtained the similar result on the nonlinear spin dynamics of an anisotropic Heisenberg ferromagnetic spin chain with octupole-dipole interaction.

Without loss of generality, we will assume γ_1 equals to 1.

By virtue of

$$\begin{aligned} |u|^2 = 1 \quad &\Rightarrow \quad u \cdot u_x = 0 \\ &\Rightarrow \quad u \cdot u_{xx} = -|u_x|^2 \\ &\Rightarrow \quad u \cdot u_{xxx} = -\frac{3}{2}\big(|u_x|^2\big)_x, \end{aligned}$$

we can rewrite (1.5) as

$$u_t = u \times u_{xxxx} + \left[\left(1 + \frac{5}{2}|u_x|^2\right)u \times u_x\right]_x. \tag{1.6}$$

We will consider the following periodic initial-value problem:

$$\begin{cases} u_t = u \times u_{xxxx} + \left[\left(1 + \frac{5}{2}|u_x|^2\right) u \times u_x\right]_x, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x+2D,t) = u(x,t), & \text{in } R \times [0,T), \end{cases} \tag{1.7}$$

where $\Omega = (-D, D)$ and D is a positive constant.

By the viscosity vanishing method, we can derive the following theorem.

Theorem 1.1. *Let $u_0 \in H^2(\Omega)$, $|u_0| = 1$ a.e. Then there exists a weak solution for* (1.7) *such that*

(i) *$u \in L^\infty(0,T; H^2(\Omega))$, $|u| = 1$ a.e.;*
(ii) *for any $\varphi \in C^\infty(\bar{\Omega} \times [0,T])$, $\varphi(x,T) = 0$, $\varphi(-D,t) = \varphi(D,t)$, there holds*

$$\begin{aligned} -\int_{\Omega\times(0,T)} u \cdot \varphi_t \, dx\, dt = \int_{\Omega} u_0 \varphi(x,0)\, dx &+ \int_{\Omega\times(0,T)} (u \times u_{xx}) \cdot \varphi_{xx}\, dx\, dt \\ &+ 2 \int_{\Omega\times(0,T)} (u_x \times u_{xx}) \cdot \varphi_x \, dx\, dt \\ &+ \int_{\Omega\times(0,T)} \left(1 + \frac{5}{2}|u_x|^2\right) u \times u_{xx} \cdot \varphi \, dx\, dt \\ &+ \int_{\Omega\times(0,T)} 5(u_x \cdot u_{xx}) u \times u_x \cdot \varphi \, dx\, dt, \end{aligned}$$

where T depends on $\|u_0\|_{H^2}$.

The paper is organized as follows: In Section 2 we consider (1.7) with an additional ε viscosity term, see (2.1); In Section 3 we first give a uniform estimate on the solution u_ε of (2.1) with respect to ε, then we take the limit $\varepsilon \to 0$ to obtain the weak solution of (1.7).

2. An ε-parameter equation

Consider the following equation:

$$\begin{cases} u_t = u \times u_{xxxx} + \left[\left(1 + \frac{5}{2}|u_x|^2\right) u \times u_x\right]_x - \varepsilon u_{xxxx} \\ \qquad - \varepsilon\left(\frac{3}{2}|u_x|^2_{xx} + u_x \cdot u_{xxx}\right) u, \\ u(x,0) = u_0, \\ u(x+2D,t) = u(x,t). \end{cases} \tag{2.1}$$

We have the following lemma about smooth solutions for (2.1).

Lemma 2.1. *If u is a smooth solution of* (2.1)*, then $|u| = 1$.*

Proof. Let us set $v(x,t)=|u(x,t)|^2$, then v satisfies

$$\begin{cases} v_t=-\varepsilon v_{xxxx}-\varepsilon\left(\frac{3}{2}|u_x|_{xx}^2+u_x\cdot u_{xxx}\right)(v-1),\\ v(x,0)=1,\\ v(x+2D,t)=v(x,t). \end{cases}$$

Furthermore we set $w=v-1$, then w satisfies

$$w_t=-\varepsilon w_{xxxx}-G(x,t)w, \tag{2.2}$$

$$w(x,0)=0, \tag{2.3}$$

$$w(x+2D,t)=w(x,t), \tag{2.4}$$

where $G(x,t)=\frac{3}{2}|u_x|_{xx}^2+u_x\cdot u_{xxx}$. Taking the scalar product with w in (2.2), and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-D}^{D}|w|^2\,dx+\varepsilon\int_{-D}^{D}|w_{xx}|^2\,dx\leqslant\varepsilon\max_{x,t}\bigl|G(x,t)\bigr|\int_{-D}^{D}|w|^2\,dx.$$

From Gronwall's inequality and $w(x,0)=0$, we deduce that $w(x,t)=0$, that is, $|u(x,t)|=1$. □

Next we use the difference method to prove the existence of solutions for (2.1).

To proceed, we need the following classical notations:

$$u_h=\bigl\{u_j=u(x_j)\bigm| j=0,1,\dots,J\bigr\},\quad x_j=jh,\ h=2D/J,$$

$$\|u_h\|_{L^\infty}=\sup_{j\in\{0,1,\dots,\ J\}}|u_j|,$$

$$\|u_h\|_p=\left(\sum_{j=0}^{J-1}|u_j|^p h\right)^{1/p},$$

$$\bigl\|\delta^k u_h\bigr\|_p=\left(\sum_{j=0}^{J-k}\left|\frac{\Delta_+^k u_j}{h^k}\right|^p h\right)^{\frac{1}{p}},$$

where $2\leqslant p<\infty$.

With these notations, we have the following lemmas.

Lemma 2.2. [8] *Let q, r be real numbers and j, m be integers such that $1\leqslant q,r\leqslant\infty$, $0\leqslant j<m$. If $u\in W^{m,r}(\Omega)\cap L^q(\Omega)$, then*

$$\bigl\|D^j u\bigr\|_p\leqslant C\|u\|_q^{1-\alpha}\|u\|_{m,r}^{\alpha},$$

where $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{m,r}=\|\cdot\|_{W^{m,r}(\Omega)}$, $p\geqslant 1$, $\frac{j}{m}\leqslant\alpha\leqslant 1$ and

$$\frac{1}{p}-j=\frac{1-\alpha}{q}+\alpha\left(\frac{1}{r}-m\right).$$

Lemma 2.3. [9] *Let p be a real number and j, m be integers such that $2 \leqslant p \leqslant \infty$, $0 \leqslant j < m$. Then*

$$\left\|\delta^j u_h\right\|_p \leqslant C\|u\|_2^{1-\alpha}\left(\left\|\delta^m u_h\right\|_2+\frac{\|u_h\|_2}{(2D)^m}\right)^\alpha,$$

where $\alpha = \frac{1}{m}(j+\frac{1}{2}-\frac{1}{p})$.

Lemma 2.4. [9] *Let $u_h = \{u_j \mid j = 0, \pm 1, \ldots, \pm J, \ldots\}$, $v_h = \{v_j \mid j = 0, \pm 1, \ldots, \pm J, \ldots\}$ such that*

$$u_{j+J} = u_j,$$
$$v_{j+J} = v_j.$$

Then

(i) $\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j$;
(ii) $\sum_{j=1}^{J} u_j \Delta_+\Delta_- v_j = -\sum_{j=0}^{J-1} \Delta_+ u_j \Delta_+ v_j$;
(iii) $\Delta_+(u_j v_j) = u_{j+1}\Delta_+ v_j + v_j \Delta_+ u_j$,

where Δ_+, Δ_- denote the forward and backward difference, respectively.

Now we can establish the following difference-differential equation corresponding to (2.1) ($\varepsilon = 1$):

$$\begin{aligned}\frac{\mathrm{d}u_j}{\mathrm{d}t} &= u_j \times \frac{\Delta_+^2\Delta_-^2 u_j}{h^4} + \left(1+\frac{5}{2}\left|\frac{\Delta_+ u_j}{h}\right|^2\right)u_j \times \frac{\Delta_+\Delta_- u_j}{h^2} \\ &\quad + 5\left(\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}\right)u_j \times \frac{\Delta_+ u_j}{h} - \frac{\Delta_+^2\Delta_-^2 u_j}{h^4} \\ &\quad - \left(3\left|\frac{\Delta_+\Delta_- u_j}{h^2}\right|^2 + 4\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+^2\Delta_- u_j}{h^3}\right)u_j, \end{aligned} \tag{2.5}$$

with

$$u_j|_{t=0} = u_{0j} = u_0(jh), \tag{2.6}$$
$$u_{j+J} = u_j, \tag{2.7}$$

where $j = 0, \pm 1, \ldots, \pm J, \ldots$, $h = 2D/J$, $J > 0$.

It is obvious that the ordinary differential equation (2.5) admits a local smooth solution. For this solution, we shall give some estimates uniformly in h, then pass to the limit $h \to 0$ to get a local smooth solution of (2.2).

Lemma 2.5. *If $u_j(x) \in H^2(\Omega)$, then there exist constants $T > 0$ and $C > 0$ independent of h such that*

$$\sup_{0\leqslant t\leqslant T} \|u_h\|_2 \leqslant C,$$
$$\sup_{0\leqslant t\leqslant T} \|\delta u_h\|_2 \leqslant C,$$
$$\sup_{0\leqslant t\leqslant T} \left\|\delta^2 u_h\right\|_2 \leqslant C.$$

Proof. (1) Taking the scalar product with $u_j h$ in (2.5) and summing up over j from 1 to J lead to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{J}|u_j|^2 h = -\sum_{j=1}^{J} u_j \frac{\Delta_+^2\Delta_-^2 u_j}{h^4} - \sum_{j=1}^{J}\left(3\left|\frac{\Delta_+\Delta_- u_j}{h^2}\right|^2 + 4\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+^2\Delta_- u_j}{h^3}\right)|u_j|^2 h. \tag{2.8}$$

Applying Lemmas 2.3, 2.4 and Young's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{J}|u_j|^2 h + \left\|\delta^2 u_h\right\|_2^2 \leqslant \frac{1}{4}\left\|\delta^3 u_h\right\|_2^2 + C\left(1 + \|u_h\|_2^{12} + \|\delta u_h\|_2^4 + \left\|\delta^2 u_h\right\|_2^4\right). \tag{2.9}$$

(2) Taking the scalar product with $(\Delta_+\Delta_- u_j)/h$ in (2.5) and summing up over j from 1 to J lead to

$$\begin{aligned}\sum_{j=1}^{J} u_{jt}\frac{\Delta_+\Delta_- u_j}{h} &= \sum_{j=1}^{J} u_j \times \frac{\Delta_+^2\Delta_-^2 u_j}{h^4}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}h \\ &\quad + 5\sum_{j=1}^{J}\left(\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}\right)u_j \times \frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}h \\ &\quad - \sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-^2 u_j}{h^4}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}h \\ &\quad - \sum_{j=1}^{J}\left(3\left|\frac{\Delta_+\Delta_- u_j}{h^2}\right|^2 + 4\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+^2\Delta_- u_j}{h^3}\right)u_j\cdot\frac{\Delta_+\Delta_- u_j}{h^2}h.\end{aligned}$$

Applying Lemmas 2.3, 2.4, Young's inequality and Hölder's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u_h\|_2^2 + \frac{3}{2}\left\|\delta^3 u_h\right\|_2^2 \leqslant C\left(1 + \|u_h\|_2^{12} + \|\delta u_h\|_2^8 + \left\|\delta^2 u_h\right\|_2^{10}\right). \tag{2.10}$$

(3) Taking the scalar product with $(\Delta_+^2\Delta_-^2 u_j)/h^3$ in (2.5) and summing up over j from 1 to J lead to

$$\begin{aligned}\sum_{j=1}^{J} u_{jt}\frac{\Delta_+^2\Delta_-^2 u_j}{h^3} &= \sum_{j=1}^{J}\left(1 + \frac{5}{2}\left|\frac{\Delta_+ u_j}{h}\right|^2\right)u_j \times \frac{\Delta_+\Delta_- u_j}{h^2}\cdot\frac{\Delta_+^2\Delta_-^2 u_j}{h^4}h \\ &\quad + \sum_{j=1}^{J}5\left(\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+\Delta_- u_j}{h^2}\right)u_j \times \frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+^2\Delta_-^2 u_j}{h^4}h \\ &\quad - \sum_{j=1}^{J}\left|\frac{\Delta_+^2\Delta_-^2 u_j}{h^4}\right|^2 h \\ &\quad - \sum_{j=1}^{J}\left(3\left|\frac{\Delta_+\Delta_- u_j}{h^2}\right|^2 + 4\frac{\Delta_+ u_j}{h}\cdot\frac{\Delta_+^2\Delta_- u_j}{h^3}\right)u_j\cdot\frac{\Delta_+^2\Delta_-^2 u_j}{h^4}h.\end{aligned}$$

Applying Lemmas 2.3, 2.4, Young's inequality and Hölder's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\delta^2 u_h\right\|_2^2+\frac{3}{4}\left\|\delta^3 u_h\right\|_2^2 \leqslant C\left(1+\|u_h\|_2^{32}+\|\delta u_h\|_2^{40}+\left\|\delta^2 u_h\right\|_2^{16}\right). \tag{2.11}$$

Adding (2.9)–(2.11) up, after a simple calculation we have

$$\left\|\delta^k u_h\right\|_2 \leqslant C, \quad \int_0^T \left\|\delta^{k+2} u_h\right\|_2^2 \leqslant C, \quad k=0,1,2,\ 0\leqslant t\leqslant T,$$

where T depends on D, $\|u_{0h}\|_2$, $\|\delta u_{0h}\|_2$ and $\|\delta^2 u_{0h}\|_2$.

By induction and a limiting procedure, we can eventually derive a local smooth solution on $[0,T]$ for (2.1). □

3. Uniform estimates and the limiting case

We have indeed proved the existence of smooth solutions u_ε for the ε-equation (2.1). In addition, we need the uniform estimates on u_ε with respect of ε. To demonstrate the method used in this section, we begin with the estimate on the solution for the original equation (1.7).

Lemma 3.1. *If u is a smooth solution for* (1.7)*, then for any $T_1>0$, there exists a constant $C>0$ depending only on $\|u_0\|_{H^2}$ such that*

$$\sup_{0\leqslant t\leqslant T_1}\|u\|_{H^2(\Omega)}\leqslant C.$$

Proof. (1) Taking the scalar product with u in (1.7), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2=0,$$

which implies that $|u|\equiv|u_0|=1$.

(2) Taking the scalar product with $-u_{xx}$ in (1.7) and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_2^2=\int_\Omega u\times u_{xx}\cdot u_{xxxx}\,dx-5\int_\Omega (u_x\cdot u_{xx})(u\times u_x\cdot u_{xx})\,dx=\mathrm{I}_1-\mathrm{I}_2,$$

$$\mathrm{I}_1=-\int_\Omega (u_x\times u_{xx})\cdot u_{xxx}\,dx. \tag{3.1}$$

Since $|u|=1$, we have $u\cdot u_x=0$. Thus u, u_x and $u\times u_x$ consist of an orthonormal basis of R^3. u_{xx} can be formulated as

$$u_{xx}=\alpha_1 u+\alpha_2 u_x+\alpha_3 u\times u_x, \tag{3.2}$$

where $\alpha_1=u\cdot u_{xx}=-|u_x|^2$, $\alpha_2=(u_x\cdot u_{xx})/|u_x|^2$ and $\alpha_3=(u\times u_x\cdot u_{xx})/|u_x|^2$.

Substituting $\alpha_1,\alpha_2,\alpha_3$ into (3.2), we have

$$u_x\times u_{xx}=|u_x|^2 u\times u_x+(u\times u_x\cdot u_{xx})u. \tag{3.3}$$

So

$$\mathrm{I}_1=-\int_\Omega (u\times u_x\cdot u_{xxx})|u_x|^2\,dx-\int_\Omega (u\times u_x\cdot u_{xx})(u\cdot u_{xxx})\,dx. \tag{3.4}$$

If we differentiate $u \cdot u_x = 0$ twice, we have

$$u \cdot u_{xxx} = -\frac{3}{2}|u_x|_x^2. \tag{3.5}$$

For the first term on the right-hand side of (3.4), we have

$$-\int_\Omega (u \times u_x \cdot u_{xxx})|u_x|^2\,dx = -\int_\Omega (u \times u_x \cdot u_{xx})_x|u_x|^2\,dx = -\int_\Omega (u \times u_x \cdot u_{xx})|u_x|_x^2\,dx. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we have

$$\mathrm{I}_1 = 5\int_\Omega (u \times u_x \cdot u_{xx})(u_x \cdot u_{xx})\,dx = \mathrm{I}_2,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_2^2 = 0 \quad \Rightarrow \quad \big\|u_x(t)\big\|_2 = \big\|u_x(0)\big\|_2, \quad 0 < t < T_1. \tag{3.7}$$

(3) Taking the scalar product with u_{xxxx} in (1.7) and integrating on Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{xx}\|_2^2 = \int_\Omega \left(1 + \frac{5}{2}|u_x|^2\right) u \times u_{xx} \cdot u_{xxxx}\,dx + 5\int_\Omega (u_x \cdot u_{xx})(u \times u_x \cdot u_{xxxx})\,dx. \tag{3.8}$$

We also have

$$\begin{aligned}
\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_4^4 &= \int_\Omega |u_x|^2 u_x \cdot u_{xt}\,dx \\
&= -\int_\Omega |u_x|^2 u_{xx} \cdot u_t\,dx - 2\int_\Omega (u_x \cdot u_{xx})(u_x \cdot u_t)\,dx \\
&= -\int_\Omega |u_x|^2 u_{xx} \cdot \big(u \times u_{xxxx} + 5(u_x \cdot u_{xx}) u \times u_x\big)\,dx \\
&\quad - 2\int_\Omega (u_x \cdot u_{xx}) u_x \cdot \left(u \times u_{xxxx} + \left(1 + \frac{5}{2}|u_x|^2\right) u \times u_{xx}\right)dx \\
&= -\int_\Omega |u_x|^2 u_{xx} \cdot (u \times u_{xxxx})\,dx - 2\int_\Omega (u_x \cdot u_{xx}) u_x \cdot (u \times u_{xxxx})\,dx \\
&\quad - 2\int_\Omega (u_x \cdot u_{xx}) u_x \cdot (u \times u_{xx})\,dx \\
&= \int_\Omega |u_x|^2 u \times u_{xx} \cdot u_{xxxx}\,dx + 2\int_\Omega (u_x \cdot u_{xx}) u \times u_x \cdot u_{xxxx}\,dx \\
&\quad - 2\int_\Omega (u_x \cdot u_{xx}) u_x \cdot (u \times u_{xx})\,dx.
\end{aligned} \tag{3.9}$$

The last term on the right-hand side of (3.9) is equal to $\frac{2}{5}\int_\Omega u \times u_{xx} \cdot u_{xxxx}\,dx$ from step (2), so comparing (3.9) with (3.8), we have

$$4\frac{\mathrm{d}}{\mathrm{d}t}\|u_{xx}\|_2^2 = 5\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_4^4. \tag{3.10}$$

Integrating (3.10) from 0 to T_1, we have

$$4\big\|u_{xx}(T_1)\big\|_2^2 - 5\big\|u_x(T_1)\big\|_4^4 = \|u_{0xx}\|_2^2 - 5\|u_{0x}\|_4^4. \tag{3.11}$$

Applying Lemma 2.2, we have

$$\|u_x\|_4^4 \leqslant C\|u_{xx}\|_2\|u_x\|_2^3. \tag{3.12}$$

From (3.7), (3.11), (3.12) and Young's inequality, we have

$$\big\|u_{xx}(T_1)\big\|_2 \leqslant C \quad \text{for any } T_1. \qquad \square$$

Now it is possible to present the uniform estimates on smooth solutions for (2.1).

Lemma 3.2. *If u_ε is a smooth solution for* (2.1) *on* $[0,T]$*, then there exist a constant $C > 0$ depending only on $\|u_0\|_{H^2}$ and T such that*

$$\big\|u_\varepsilon(t)\big\|_{H^2(\Omega)} \leqslant C, \quad 0 \leqslant t \leqslant T.$$

Proof. (1) From Lemma 2.1 we infer that $|u_\varepsilon| = 1$.

(2) As the same procedure as step (2) in Lemma 3.1, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_{\varepsilon x}|^2\,dx = \varepsilon\int_\Omega |u_{\varepsilon xxx}|^2\,dx - 3\varepsilon\int_\Omega |u_{\varepsilon xx}|^2 u_\varepsilon \cdot u_{\varepsilon xx}\,dx - 4\varepsilon\int_\Omega (u_{\varepsilon xx}\cdot u_{\varepsilon xxxx})(u_\varepsilon\cdot u_{\varepsilon xx})\,dx.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon x}\|_2^2 + \varepsilon\|u_{\varepsilon xxx}\|_2^2 \leqslant 3\|u_{\varepsilon xx}\|_3^3 + 4\varepsilon\int_\Omega |u_{\varepsilon x}||u_{\varepsilon xx}||u_{\varepsilon xxx}|\,dx. \tag{3.13}$$

Applying Lemma 2.2, we obtain

$$\|u_{\varepsilon xx}\|_3 \leqslant C\|u_{\varepsilon x}\|_2^{\frac{5}{12}}\big(\|u_{\varepsilon xxx}\|_2 + \|u_{\varepsilon x}\|_2\big)^{\frac{7}{12}},$$

and

$$\|u_{\varepsilon xx}\|_2 \leqslant C\|u_{\varepsilon x}\|_2^{\frac{1}{4}}\big(\|u_{\varepsilon xxx}\|_2 + \|u_{\varepsilon x}\|_2\big)^{\frac{3}{4}}.$$

Substituting the above inequalities into (3.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon x}\|_2^2 + \frac{\varepsilon}{2}\|u_{\varepsilon xxx}\|_2^2 \leqslant C\varepsilon\big(1 + \|u_{\varepsilon x}\|_2^{10}\big).$$

When ε is small enough, we have

$$\big\|u_{\varepsilon x}(t)\big\|_2 \leqslant C, \tag{3.14}$$

where $0 \leqslant t \leqslant T$.

(3) As the same procedure as step (3) in Lemma 3.1, we have

$$\begin{aligned}
&4\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon xx}\|_2^2-5\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon x}\|_4^4+8\varepsilon\int_\Omega|u_{\varepsilon xxxx}|^2\,dx\\
&\quad\leqslant C\varepsilon\int_\Omega|u_\varepsilon\cdot u_{\varepsilon xxxx}|^2\,dx+C\varepsilon\int_\Omega|u_{\varepsilon x}|^2|u_{\varepsilon xx}\cdot u_{\varepsilon xxxx}|\,dx\\
&\qquad+C\varepsilon\int_\Omega|u_{\varepsilon x}|^2(u_{\varepsilon xx}\cdot u_\varepsilon)(u_\varepsilon\cdot u_{\varepsilon xxxx})\,dx.
\end{aligned}$$

Applying Lemma 2.2, we obtain

$$\|u_{\varepsilon xx}\|_\infty\leqslant C\|u_{\varepsilon x}\|_2^{\frac12}\big(\|u_{\varepsilon xxxx}\|_2+\|u_{\varepsilon x}\|_2\big)^{\frac12},$$

and

$$\|u_{\varepsilon x}\|_4\leqslant C\|u_{\varepsilon xx}\|_2^{\frac{15}{16}}\big(\|u_{\varepsilon xxxx}\|_2+\|u_{\varepsilon x}\|_2\big)^{\frac{1}{16}}.$$

By virtue of (3.14), Young's inequality and Hölder's inequality, we have

$$4\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon xx}\|_2^2-5\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon x}\|_4^4\leqslant C\varepsilon. \tag{3.15}$$

Integrating (3.15) from 0 to t, we have

$$4\big\|u_{\varepsilon xx}(t)\big\|_2^2-4\big\|u_{\varepsilon xx}(0)\big\|_2^2-5\big\|u_{\varepsilon x}(t)\big\|_4^4+5\big\|u_{\varepsilon x}(0)\big\|_4^4\leqslant C\varepsilon. \tag{3.16}$$

By (3.12), we infer from (3.16) that

$$\big\|u_{\varepsilon xx}(t)\big\|_2\leqslant C,\quad 0\leqslant t\leqslant T. \qquad\square$$

Next we shall deal with the limiting case. Lemma 3.2 and Theorem 5.1 of Chapter 1 [10] imply that there exists a subsequence of u_ε (still denoted by u_ε) such that when $\varepsilon\to0$, there holds

$$u_\varepsilon\to u\quad\text{strongly in } L^p\big(\Omega\times(0,T)\big), \tag{3.17}$$

and

$$u_{\varepsilon x}\to u_x\quad\text{strongly in } L^p\big(\Omega\times(0,T)\big), \tag{3.18}$$

where $1\leqslant p<\infty$.

Furthermore, we have

$$u_{\varepsilon xx}\to u_{xx}\quad\text{weakly* in } L^\infty\big(0,T;L^2(\Omega)\big). \tag{3.19}$$

With the help of these convergence results, taking the scalar product with test function φ in (2.1) and integrating on $\Omega\times(0,T)$, we conclude Theorem 1.1 from the integration by parts.

References

[1] M. Lakshmanan, K. Porsezian, M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, Phys. Lett. A 133 (1988) 483–488.

[2] K. Porsezian, M. Daniel, M. Lakshmanan, On the integrability aspects of the one-dimensional classical continuum isotropic Heisenberg spin chain, J. Math. Phys. 33 (5) (1992) 1807–1816.

[3] L.D. Landau, E.M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowjetunion 8 (1935), reproduced in: D. Ter Hear (Ed.), Collected Papers of L.D. Landau, Pergamon, New York, 1965, pp. 101–114.

[4] J. Fivez, On the continuum limit of a classical compressible Heisenberg chain, Solid State Phys. 15 (1982) 641–643.

[5] M. Cieplak, L.A. Turski, Magnetic solitons and elastic kink-like excitations in compressible Heisenberg chain, J. Phys. C (Solid State) 13 (1980) L777–L780.

[6] S. Ding, B. Guo, F. Su, Measure-valued solution to the compressible Heisenberg chain equations, J. Math. Phys. 40 (3) (1999) 1153–1162.

[7] M. Daniel, L. Kavitha, R. Amuda, Soliton spin excitations in an anisotropic Heisenberg ferromagnet with octupole-dipole interaction, Phys. Rev. B 59 (21) (1999) 13774–13781.

[8] Yulin Zhou, Boling Guo, Shaobin Tan, Existence and uniqueness of smooth solution for system of ferromagnetic chain, Sci. China Ser. A 34 (1991) 257–266.

[9] Y.L. Zhou, Interpolation formulas of intermediate quotients for discrete functions with several indices, J. Comput. Math. 2 (1984) 276–281.

[10] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaores, Dunod/Gauthier-Villars, Paris, 1969.